C. MØLLER

ON THE CRISIS IN THE THEORY OF GRAVITATION AND A POSSIBLE SOLUTION

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **39**, 13

Kommissionær: Munksgaard
København 1978

Synopsis

It is now generally believed that *Einstein's* beautiful *theory of gravitation* under special circumstances leads to inconsistent results. In fact, according to this theory a well-defined physical system may after a finite time pass over into an unphysical state, where the metric is singular and consequently the notions of space and time lose their physical meaning. This inconsistency calls for a generalized theory of gravitation for macroscopic matters which is free of singularities and at the same time retains all the satisfactory features of Einstein's theory. It is shown that such a generalization may possibly be obtained by assuming that the fundamental gravitational variables are, not the metric tensor, but the components of a tetrad field from which the metric of space-time can be derived uniquely. In a *tetrad theory of gravitation* the basic principles of Einstein's theory are still valid exactly, first of all the principle of general relativity, the principle of equivalence, and the fusion of gravity and mechanics. Such a theory also leads to a more satisfactory solution of the energy problem.

Printed in Denmark by Bianco Lunos Bogtrykkeri A/S. ISBN 87-7304-092-4

1. Statement of the Problem

During the last two decades all the effects predicted by Einstein's theory of general relativity and gravitation (EGRG) have been experimentally verified with a reasonably high degree of accuracy. It is true that these tests are concerned with cases only where the gravitational field is comparatively weak; but the simplicity and generality of the principles underlying the theory as well as its intrinsic consistency and cogency made it reasonable to assume that the theory be valid for stronger fields also.

However, at the same time investigations concerning the stability of large amounts of mass led to strange results which implied a serious crisis for EGRG, or for physics itself if this theory is taken for gospel truth. In fact it was shown[1)] that a sufficiently large amount of matter according to EGRG will undergo a steady contraction under the influence of its own gravitational field. After a finite time as measured on a standard clock following the matter, the system is engulfed in a 'black hole' from which no message can be sent into the outside world, and after a further very short time the system collapses into a singularity, where not only the mass density is infinite, but where the space-time metric itself becomes singular.

Thus, according to Einstein's theory a well-defined physical system may after a finite time pass over into an unphysical state, where the notions of space and time become meaningless. Since these notions enter in an essential way in the formulation of all physical laws this means the breakdown of physics; for one cannot know what will come out of a singularity, and it is then not possible any more to predict the future.

For a long time many physicists (including myself) did not believe that Einstein's otherwise so usccessful theory had such disastrous consequences[2)]; but by now there seems to be a consensus of opinion that these space-time singularities are inevitable, whenever the energy-momentum tensor, which in Einstein's theory represents the source of the gravitational field, satisfies certain physically reasonable conditions. Some physicists have tried to maintain that the situation is not so bad; for the final collapse of the system into the singularity is preceded by its passage through the Schwarzschild wall (the event horizon) that delimits the black hole, and in this state no light or any other signal from the system can penetrate into the outside world, so that the final collapse is totally unobservable from outside. Moreover, an observer at a constant distance r outside the Schwarzschild wall with radius α will strictly speaking never experience the formation of the black hole; for measured on a standard clock at rest at constant $r > \alpha$ the formation of a black hole will take an infinite time, in contrast to the finite time as measured by a standard clock following the matter. (An extreme example of the relativity of time.) However, this attempt of explaining away the difficulty is not very satisfactory. What about observers that are sitting on the collapsing matter, should the laws of physics not be valid for them? Was it not just one of the main requirements of general relativity that these laws should be of the same form for arbitrarily moving observers?

Other physicists hope that a quantization of the metric field along the lines followed in quantum electrodynamics could prevent the collapse into the singularity, similarly as the introduction of Planck's quantum of action into mechanics and electrodynamics prevents the collapse of the Rutherford model of the atom. Indeed it would seem reasonable to expect quantum gravitational effects to be important for the very strong fields in the small regions of space-time in the vicinity of a singularity. However, in the first place it does not seem possible to carry through the quantization program for the gravitational field along the same lines as in quantum electrodynamics, because the non-linear gravitational field of general relativity is basically non-renormalizable. Moreover the root of the trouble does not seem to lie exclusively in the very small regions near the singularity, but rather in the whole usually macroscopic domain of the black hole.

In a number of interesting papers Hawking[3)], Wald[4)] and Parker[5)] have shown that black holes create and emit particles at a steady rate. It is maintained that this radiation will cause the black hole to lose mass and eventually to disappear, leaving a naked singularity behind. In this situation there is a basic limitation on our ability to predict the future, which Hawking[6)] has formulated in a new physical principle—the randomicity principle. According to this principle all configurations for particles emitted from a black hole singularity compatible with the external constraints are equally probable. This means that a complete set of data on a space-like surface is not sufficient in general to determine with certainty the behaviour of a system, since information may disappear into or suddenly appear from a hole singularity.

The randomicity principle implies a much more radical departure from the deterministic description of classical physics than that which was brought about by the principles of quantum mechanics. In the latter theory it was recognized that the deterministic Newtonian equations of mechanics could not be used to predict the motion of an electron exactly, because this would presuppose that we can know the initial position and momentum of the electron exactly, which is impossible according to Heisenberg's uncertainty principle. On the other hand, the randomicity principle claims that the future state in certain cases may be undetermined even if the initial state is well-defined, which would make physics truly indeterministic.

This is such a serious departure from the philosophy, which has been the mainstay of physics since Galileo, that many physicists will ask if this step is really necessary. Could it not be that Einstein's classical theory of gravitation, on which Hawking's conclusions are based, breaks down in the case of very strong gravitational fields. After all the theory has been experimentally verified for comparatively weak fields only, and surely Einstein's theory like all other theories must be expected to have a limited domain of applicability. In fact, in the past the occurrence of essential singularities in a physical theory has usually been taken as a sign that the theory has been applied in a region that lies outside its domain of applicability.

As an example let us recall the situation concerning the black body radiation which caused Max Planck so much trouble around 1900. If one applies the laws of classical physics in calculating the energy density of the radiation inside a cavity in thermal equilibrium, one obtains the formula of Rayleigh-Jeans, according to which the energy density per unit frequency interval is proportional to the square of the frequency ν. Thus the total energy density, obtained by integrating over all ν, is infinite which obviously is meaningless. This "ultra-violet catastrophe" indicates that we have applied the laws of classical physics to a phenomenon that lies outside their domain of applicability. Using instead the laws of quantum physics, that are valid also for large ν, we are led to Planck's formula for the energy density which gives finite results.

Similarly one would be inclined to think that the occurrence of essential singularities in Einstein's theory indicates that this theory breaks down in the case of very strong gravitational fields—a thought that was not unfamiliar to Einstein himself[7]. This point of view is supported by the circumstance that EGRG actually ceases to be a physical theory connecting measurable physical quantities already before the system passes into the singularity. In order to measure the metric, for instance, we need an instrument which measures the proper time, i.e. a physical clock which shows the same time as the ideal standard clocks with which one operates in general relativity[8]. It is well-known that an oscillatory system with atomic frequency represents an extremely good standard clock in ordinary gravitational fields. However, as was shown in a recent paper[9] any such clock ceases to give the correct proper time when approaching and before actually reaching a singularity. For this reason we concluded that the proper time and therefore also the metric itself lose their physical meaning already somewhat outside the singularities in question.

Under these circumstances it seems imperative to investigate the possibility of constructing a theory of gravitation for macroscopic matter that is free of singularities and at the same time retains all the satisfactory features of EGRG. According to the preceding discussion this would presumably have to be a theory in which there are no black holes and which gives the same results as Einstein's theory at least for weak fields up to the second order of approximation. However we would have to require more than just that; for there can be no question of returning to the ideas prevailing in physics before 1915. A number of the principles on which Einstein based his theory must be regarded as irrevocable.

In the following we have listed the most fundamental assumptions and properties of EGRG which it would be desirable to retain in a generalized theory:

A. *Space-time is a manifold with a pseudo-Riemannian metric. The metric tensor* g_{ik} *is a physical quantity that can be measured in principle by means of standard clocks, and the determinant* $g = \det(g_{ik})$ *is everywhere negative:*

$$g < 0. \tag{1.1}$$

All physical laws are expressed by equations that are covariant or form-invariant under arbitrary transformations of the space-time coordinates.

In these equations the measurable quantities g_{ik} enter in an essential way along with the other physical quantities that describe the phenomena in question. The form-invariance of the equations is the mathematical expression of the general principle of relativity, according to which the fundamental laws of nature, obtained by experiments, are of the same form irrespective of the state of motion of the observers. Thus, for the first time in the history of physics a given set of phenomena is described by a uniquely determined set of equations. This inalienable property can be regarded as the crowning touch of a long development of physics from Aristotle over

Galileo and Newton to Einstein—a development that is characterized by a constantly increasing symmetry or form-invariance of the laws of nature under ever wider groups of transformations.

B. *Another precious acquisition of EGRG is the fusion of gravitation and mechanics. Einstein's gravitational field equations do not only determine the gravitational field for a given matter distribution, but also the motion of the matter source is determined by these equations: the mechanieal equations of motion are consequences of the field equations.*

For incoherent matter the equations of motion of an infinitesimal piece of matter following from the field equations are identical with the equations of motion of a freely falling test particle.

C. A basic assumption in EGRG is *the equivalence principle, according to which the effects of a gravitational field can be 'transformed away' in an infinitesimal region around a given event point P by introducing a system of coordinates that is geodesic at P. Moreover, if this system is locally Lorentzian, all the physical laws at P are of the same form as in special relativity.*

As an immediate consequence of this principle gravity must effect the trajectories of all freely moving particles in exactly the same way independently of the mass of the particle. In the case of the gravitational field of the earth this has now been verified experimentally to the very high accuracy of 10^{-11} by Dicke[10)] and Bragnisky[11)] and their co-workers. Thus at least for weak gravitational fields this consequence of the principle of equivalence can be regarded as well established.

For a matter system with the energy-momentum tensor T_{ik} it follows from the principle of equivalence that the 'conservation laws' in a general system of coordinates must be of the form

$$T_i{}^k{}_{;k} = 0 \tag{1.2}$$

where $;k$ denotes the covariant derivative formed by means of the Christoffel symbols corresponding to the metric tensor g_{ik}. Thus according to **B** the equations (1.2) must be consequences of the field equations.

Further it follows from **C** that the world line of a freely falling particle is a geodesic in the 4-space with the metric tensor g_{ik}.

D. *The gravitational field equations are derivable from a Lagrangean principle* with a Lagrangean density which is a scalar density under the group of general coordinate transformations. In this way the general covariance and the compatibility of the field equations are secured.

E. *The field equations are partial differential equations in the field variables of not higher than the second order.* This is essential for obtaining a Cauchy problem of the usual kind.

F. *In Einstein's theory the gravitational field is assumed to be exhaustively described by the metric tensor g_{ik} alone.*

According to **D** and **F** the gravitational part of the Lagrangean integral is of the form $\int L \sqrt{-g}\, dx$, where L is a scalar constructed from the g_{ik} and their derivatives. Among the numerous independent scalars of this type, the curvature scalar R plays a special role. In fact, only with $L = R$ do we get field equations of the type **E**. Thus the assumptions **A**–**F** lead uniquely to EGRG, with the field equations

$$G_{ik} = -\varkappa T_{ik}. \tag{1.3}$$

The Einstein tensor G_{ik} is a function of the g_{ik} and their space-time derivatives up to the second order and T_{ik} is the energy-momentum tensor of the matter source, which depends on g_{ik} as well as on the matter variables. On account of the Bianchi identities the divergence of the Einstein tensor vanishes identically, i.e.

$$G_i{}^k{}_{;k} \equiv 0.$$

Hence the "conservation laws" (1.2) are consequences of the field equations in accordance with **B** and **C**.

For incoherent matter we have

$$T_i^k = \mu_0 U_i U^k, \tag{1.4}$$

where μ_0 is the proper mass density and U^i is the four-velocity of the matter. With this expression for the energy-momentum tensor the equations (1.2) yield

$$(\mu_0 U^k)_{;k} = 0 \tag{1.5}$$

and

$$\frac{DU_i}{d\tau} \equiv U_{i;k} U^k = 0. \tag{1.6}$$

(1.5) expresses the conservation of proper mass, while (1.6) shows that the world line of a particle in the incoherent matter is a geodesic, as it should be according to **B** and **C** since the particle is freely falling.

The remarkable wholeness of EGRG makes a generalization of this theory a difficult job. At least it would obviously be necessary to give up some of the assumptions contained in **A**–**F**. The properties **A**, **B**, **D** and **E** are so essential that they hardly can be abandoned and also **C** seems indispensable. The equivalence principle is well established at least for weak gravitational fields. Then remains a possible change of assumption **F**.

We have already mentioned that EGRG is the only possible theory if we assume that the gravitational field is described exclusively by the metric tensor g_{ik}. Therefore we shall tentatively assume that there are as yet undiscovered properties of the gravitational field which cannot be described by the metric field only. Thus besides the g_{ik}, that certainly describe the field correctly for weak fields, we introduce additional field variables that play a role for strong gravitational fields only. The most primitive assumption is that the new gravitational field variables are independent tensor fields embedded in the Riemannian space with the metric g_{ik}. However, as we shall see now, this does not work.

Let us consider the case of an antisymmetric tensor field Γ_{ik} which satisfies equations of the same form as the Maxwell equations in general relativity, but with the electric rest charge density replaced by the proper mass density* multiplied by a new universal constant λ. Then the formalism is entirely analogous with the Einstein-Maxwell equations for electrically charged matter. The field equations for the metric tensor will be influenced by the presence of the Γ-field since the energy-momentum tensor of the latter field will act as an extra source along with the energy-momentum tensor of the matter. From **B** it follows then that a "freely falling" particle of proper mass m_0 is acted upon by a gravitational four-force

$$k_i = \lambda m_0 \Gamma_{ik} U^k / c \tag{1.7}$$

on the analogy of the electromagnetic Lorentz force.

The extra gravitational force between two massive bodies following from (1.7) is repulsive, independent of the sign of λ, and it increases indefinitely with decreasing distance, which might help preventing a gravitational collapse. On the other hand, the presence of the force k_i means that the equivalence principle **C** is not exactly valid. In a locally Lorentzian system of coordinates the gravitational field is not completely transformed away. However for sufficiently small λ, **C** may still be approximately valid for weak gravitational fields.

The solutions of the metric field equations are in this case quite analogous with the solutions of the Einstein equations given by Reissner[12)] and Weyl[13)] for the electromagnetic case. In the empty space outside a spherically sym-

* Strictly speaking this is possible for incoherent matter only. In the general case the proper mass has to be replaced by the conserved "bare mass'.

metric distribution of matter we have therefore in a system of "curvature coordinates" $\{r, \theta, \varphi, ct\}$:

$$ds^2 = adr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - bc^2dt^2 \tag{1.8}$$

with

$$b = \frac{1}{a} = 1 - \frac{\alpha}{r} + \frac{\beta^2}{r^2}. \tag{1.9}$$

Here the constant α is approximately equal to the Schwarzschild radius, i.e.

$$\alpha = \frac{\varkappa Mc^2}{4\pi} \tag{1.10}$$

and for the constant β we get approximately

$$\beta^2 = \alpha^2 \frac{\lambda^2}{2\varkappa c^4}. \tag{1.11}$$

Instead of the single event horizon in the Schwarzschild solution, we have in (1.8) two horizons in general, viz. at the values of r for which $b = 0$. a and b will be everywhere positive only when $\beta^2/\alpha^2 > \frac{1}{4}$, or by (1.11), when the dimensionless quantity $\lambda^2/\varkappa c^4$ satisfies the condition

$$\frac{\lambda^2}{\varkappa c^4} > \tfrac{1}{2} \tag{1.12}$$

in which case black holes would be excluded. However, in order to have agreement with EGRG and the experiments in the case of weak fields in particular as regards the red shift effect, it can be shown that $\lambda^2/\varkappa c^4$ cannot be larger than 0.005, i.e.

$$\lambda^2/\varkappa c^4 \ll 1. \tag{1.13}$$

Since (1.13) is in contradiction with (1.12), the introduction of the Γ-field does not solve our problem.

Let us now consider the case where the extra gravitational field is described by a scalar Ψ with field equations

$$\Gamma^i{}_{;i} = \lambda\mu_0, \qquad \Gamma_i = -\frac{\partial\Psi}{\partial x^i}. \tag{1.14}$$

Here again λ denotes a coupling constant and μ_0 is the proper mass density. In this case we have instead of (1.7) a gravitational four-force

$$k_i = -\lambda m_0 \Gamma_i. \tag{1.15}$$

The corresponding extra gravitational force between two particles is attractive and increases indefinitely with decreasing distance. Therefore there is not much hope of avoiding the singularities of EGRG in this way.

However, a suitable combination of the fields Γ_{ik} and Γ_i seems to be promising. If the coupling constants λ of the two fields are equal we have instead of (1.7) and (1.15)

$$k_i = \lambda m_0 (\Gamma_{ik} U^k / c - \Gamma_i). \tag{1.16}$$

In the Newtonian approximation, i.e. for weak fields and small velocities, it can be shown that the two terms in (1.16) cancel, so that the theory is in accordance with the results of classical celestial mechanics, even if $\lambda^2/\varkappa c^4$ is of order 1. The extra force on a particle at rest in the field of a spherical distribution of matter vanishes for large distances r, but for decreasing r this force is increasing and repulsive, so that there is a hope of avoiding collapse with this combination of fields.

A closer investigation of the solution of the metric field equations in the static spherically symmetric case shows that the conditions for the absence of event horizons is again approximately given by (1.12), which in this case is compatible with classical celestial mechanics in the Newtonian approximation. However if we go to the next approximation and consider the perihelion precession of planets, the theory gives a formula for the precession $\Delta\varphi$ that deviates from the expression $\Delta\varphi_E$ in Einstein's theory by a factor $(1-\lambda^2/\varkappa c^4)$:

$$\Delta\varphi = \Delta\varphi_E (1 - \lambda^2/\varkappa c^4). \tag{1.17}$$

Thus even with the lowest value of $\lambda^2/\varkappa c^4$ compatible with (1.12) we get a value for the perihelion precession in distinct disagreement with the observations.

Another serious difficulty is the following fact. The four-force (1.16) is not a true mechanical force of the Minkowski type[14)], since

$$k_i U^i = -\lambda m_0 \Gamma_i U^i = \lambda m_0 \frac{d\Psi}{d\tau} \neq 0. \tag{1.18}$$

This means that the proper mass m_0 of a particle is not constant in a gravitational field. Indeed from the equations of motion of a freely falling particle

$$\frac{D m_0 U_i}{d\tau} = k_i \tag{1.19}$$

we get

$$\frac{dm_0}{d\tau} = -\frac{1}{c^2} k_i U^i = -\frac{\lambda m_0}{c^2}\frac{d\Psi}{d\tau}. \tag{1.20}$$

The solution of (1.20) is

$$m_0 = m_0(0)\, e^{-\frac{\lambda\Psi}{c^2}} \tag{1.21}$$

where $m_0(0)$ is the proper mass for $\Psi = 0$ i.e. in a system of inertia.

Thus the value of the proper mass of an electron, for instance, varies with the scalar gravitational potential Ψ. Therefore also the standard frequency of a transition in an atom depends on Ψ and this dependence may even be different in atoms of different constitution. The shift of spectral lines arising from this effect has to be added to the Einstein shift. In the gravitational field of the sun or the earth, and with a λ satisfying (1.12), $\lambda\Psi$ is of the same order of magnitude as the Newtonian potential χ, so that this new effect should have been noticed in the experiments of Pound[15] and collaborators, by which Einstein's formula was verified with a high degree of accuracy.

If one goes to more complicated tensor fields than the Γ_{ik} and Γ_i it seems that it is not even possible to maintain **B**. It was bad enough that **C** could be satisfied approximately only in the just treated cases, but it would seem quite out of question ever to give up **B**. Therefore we have come to the conclusion that a generalization of Einstein's theory in accordance with known facts cannot be obtained by assuming that the metric quantities g_{ik} together with independent tensor fields are the basic gravitational field variables.

These results seem to indicate that EGRG is the only possible theory of gravitation and that the breakdown of physics referred to in the introduction is inevitable. However there is a remaining possibility in assuming that the g_{ik} are not among the truly fundamental gravitational variables, but that the latter are a set of tensor variables from which the metric quantities can be derived uniquely. Such a set of 16 independent variables are the components of so-called tetrad vector fields which determine the 10 metric components g_{ik} by simple algebraic relations.

In a paper[16] from 1961 it was shown that a tetrad description of gravitational fields also allows a more rational treatment of the energy-momentum complex than in a theory based on the metric tensor alone. In 1963 Pellegrini and Plebanski[17] gave a Lagrangean formulation of the theory and a paper[18) from 1966 contains a survey of all the investigations on the energy-momentum complex in general relativity.

The advantage of using tetrads as gravitational variables is connected with the fact that this allows to construct expressions for the energy-momentum complex which have more satisfactory transformation properties than in a purely metric formulation. However in the just mentioned investigations the admissible Lagrangeans were limited by the assumption that the equations determining the metric tensor should be exactly equal to the field equations of Einstein. In the present situation, where we are looking for metric field equations which deviate from Einstein's field equations in the case of strong gravitational fields, a wider class of Lagrangeans are admissible. In the following sections we shall see that this freedom can be used to construct a consistent theory of gravitation in which all the important properties **A**–**E** are retained and which deviates from Einstein's theory in the case of strong fields only.

2. The Basic Notions in a Tetrad Theory of Gravitation

In this section we shall give a survey of the basic notions of tetrad theories already contained in the paper reference 16, to which we shall frequently refer in what follows (the reader is requested to disregard § 6 in ref. 16).

At the out-set, before anything is filled into it, space-time is assumed to be just a continuum of points with arbitrary coordinates (x^i) but without any geometrical properties. A gravitational field in this space is described by four independent *contravariant* vector fields $\underset{a}{h^i}(x)$. Here $a = 1,2,3,4$ is an index numerating the four vectors and $i = 1,2,3,4$ is a contravariant vector index, which means that the $\underset{a}{h^i}$ transform as the coordinate differentials dx^i under all coordinate transformations. There are thus sixteen independent gravitational field variables in this theory in contrast to the ten g_{ik} in EGRG.

Consider the determinant

$$h = det(\underset{a}{h^i}) \tag{2.1}$$

with the element $\underset{a}{h^i}$ in the a'th row and the i'th column. We shall assume that this determinant is nowhere zero, i.e.

$$h \neq 0. \tag{2.2}$$

Then we can define a new set of sixteen variables $\overset{a}{h_i}$ by the equations

$$\overset{a}{h}_i \underset{b}{h}^i = \delta^a_b = (\text{Kronecker symbol}). \tag{2.3}$$

The solutions of these equations are obviously the components of four *covariant* vectors. If $\overset{a}{M}_i$ is the conjugate minor of the element $\underset{a}{h}^i$ in the determinant (2.1) the solutions of the equations (2.3) are

$$\overset{a}{h}_i = \overset{a}{M}_i / h. \tag{2.4}$$

Therefore we also have

$$\underset{a}{h}^i \overset{a}{h}_k = \delta^i_k = (\text{Kronecker symbol}). \tag{2.5}$$

From (2.3) we get, using a well-known theorem from the theory of determinants,

$$det(\underset{a}{h}^i)\cdot det(\overset{a}{h}_i) = 1, \tag{2.6}$$

where $det(\overset{a}{h}_i)$ is the determinant with $\overset{a}{h}_i$ in the a'th row and the i'th column.

Let ε_a be a quantity with components

$$\varepsilon_\alpha = 1, \qquad \alpha = 1,2,3, \qquad \varepsilon_4 = -1, \tag{2.7}$$

equal to the diagonal elements in the constant Minkowski matrix $\eta_{ab} = \eta^{ab}$, i.e.

$$\eta_{ab} = \eta^{ab} = \varepsilon_{a)} \delta^a_b, \tag{2.8}$$

where the parenthesis in a) indicates no summation over a although it appears twice in the expression on the right hand side of (2.8). Now we define two sets of vectors $\overset{a}{h}^i$ and $\underset{a}{h}_i$ by

$$\left.\begin{aligned} \overset{a}{h}^i &= \eta^{ab} \underset{b}{h}^i = \varepsilon_{a)} \underset{a}{h}^i \\ \underset{a}{h}_i &= \eta_{ab} \overset{b}{h}_i = \varepsilon_{a)} \overset{a}{h}_i \end{aligned}\right\} \tag{2.9}$$

and the inverse relations

$$\underset{a}{h}^i = \varepsilon_{a)} \overset{a}{h}^i, \quad \overset{a}{h}_i = \varepsilon_{a)} \underset{a}{h}_i, \tag{2.10}$$

i.e. the tetrad indices $a,b, \ldots$ are lowered and raised by means of the Minkowski matrix.

The presence of a gravitational field $\underset{a}{h}^i(x)$ endows the space-time continuum with definite geometrical properties. In the first place we can define a metric in this space with a metric tensor

$$g_{ik} = \underset{a}{h_i}\overset{a}{h_k} = \varepsilon_a \underset{a}{h_i}\underset{a}{h_k} = g_{ki}, \tag{2.11}$$

which obviously is a symmetric covariant tensor. Its determinant $g = det(g_{ik})$ is

$$g = det(\underset{a}{h_i})\cdot det(\overset{a}{h_k}) = -\frac{1}{h^2} \tag{2.12}$$

on account of (2.6), (2.1) and the relation

$$det(\underset{a}{h_i}) = -det(\overset{a}{h_i}) \tag{2.13}$$

following from (2.9) and (2.7). According to (2.12) and (2.2) g is always negative which means that the metric of space-time defined by (2.11) is pseudo-Riemannian like in EGRG. By a suitable choice of coordinates $\mathring{x}^i$ it is then always possible to make the values of $\mathring{g}_{ik}$ and their first order derivation at a given event point P equal to the values in a local Lorentzian system of coordinates:

$$\mathring{g}_{ik}(P) = \eta_{ik}, \;\; \mathring{g}_{ik,l}(P) = 0. \tag{2.14}$$

From (2.11) and (2.3) we get

$$g_{ik}\underset{a}{h^k} = \underset{b}{h_i}\overset{b}{h_k}\underset{a}{h^k} = \underset{b}{h_i}\delta^b_a = \underset{a}{h_i} \tag{2.15}$$

which shows that $\underset{a}{h_i}$ and $\underset{a}{h^i}$ are the covariant and contravariant components, respectively, of one and the same tetrad vector. The contravariant components of the metric tensor are then

$$g^{ik} = \underset{a}{h^i}\overset{a}{h^k} = \varepsilon_a \underset{a}{h^i}\underset{a}{h^k}. \tag{2.16}$$

Tensor indices are raised and lowered by means of the metric tensor. For a given metric the curvature of space-time can be defined as in EGRG and, as already mentioned, the only usable invariant which can be constructed from the g_{ik} and their derivatives is the curvature scalar R.

However the gravitational field $\underset{a}{h^i}$ endows space-time with other geometrical properties besides curvature viz. those connected with the notion of torsion. Thus it is not a simple Riemannian space but rather a space of the type considered first by Weizenböck[19]. If we multiply (2.3) by ε_a we get

$$\underset{a}{h_i}\underset{b}{h^i} = \varepsilon_a \delta^a_b = \eta_{ab}, \tag{2.17}$$

which shows that the four vectors $\underset{a}{h}^i$ are mutually orthogonal unit vectors in the space with the metric g_{ik}. The vectors $\underset{\alpha}{h}^i$ ($\alpha = 1,2,3$) have positive norm and are called space-like while the norm of the "time-like" vector $\underset{4}{h}^i$ is -1. Thus space-time can be pictured as a pseudo-Riemannian space with a built-in tetrad lattice.

Since space-time is more general here than in EGRG we can form a larger number of tensors and invariants. In the first place we can form the tensor

$$\gamma_{ikl} = \overset{a}{h}_i \underset{a}{h}_{k;l} = \underset{a}{h}_i \overset{a}{h}_{k;l} = -\gamma_{kil}. \tag{2.18}$$

Here $\underset{a}{h}_{k;l}$ is the usual covariant derivative of the vector $\underset{a}{h}_k$, i.e.

$$\underset{a}{h}_{k;l} = \underset{a}{h}_{k\,l} - \underset{a}{h}_r \Gamma^r_{kl}, \tag{2.19}$$

where Γ^i_{kl} is the Christoffel symbol corresponding to the metric g_{ik}. The antisymmetry in the indices i and k follows from the vanishing of the covariant derivative of g_{ik}:

$$0 = g_{ik;l} = \underset{a}{h}_{i;l} \overset{a}{h}_k + \underset{a}{h}_i \overset{a}{h}_{k;l} = \gamma_{kil} + \gamma_{ikl}. \tag{2.20}$$

Obviously γ_{ikl} is a homogeneous linear function of the first order partial derivatives of the tetrad vectors. In fact one has (see ref. 16, B.1, A.11 and A.15)

$$\gamma_{ikl} = \tfrac{1}{2} P_{ikl}{}^{rst} \overset{a}{h}_r \underset{a}{h}_{s,\,t} = -\tfrac{1}{2} P_{iklr}{}^{st} \underset{a}{h}_s \overset{a}{h}^r{}_{,\,t} \tag{2.21}$$

where

$$P_{ikl}{}^{rst} = \delta^r_i g_{kl}{}^{st} + \delta^r_k g_{li}{}^{st} - \delta^r_l g_{ik}{}^{st} \tag{2.22}$$

and

$$g_{kl}{}^{st} = \delta^s_k \delta^t_l - \delta^s_l \delta^t_k \tag{2.23}$$

are tensors that do not depend on the derivatives of the tetrad vectors. The same holds for the coefficients of $\underset{a}{h}_{s,t}$ and of $\overset{a}{h}^r{}_{,t}$ in (2.21). The tensor γ_{ikl} is closely related to the Ricci rotation coefficients (ref. 16, 3.8) and to the torsion (ref. 16, 5.14, 5.15).

A space of the Weitzenböck type has teleparallelism (ref. 16, § 5). Two vectors at distant points P_1 and P_2 may be defined as parallel when they have equal components relative to the tetrad lattice. This leads to a new type of parallel displacement and covariant differentiation of vectors with an affine connection

$$\Delta^i{}_{kl} = \overset{a}{h}{}^i \underset{a}{h}_{k,l}. \tag{2.24}$$

It differs from the usual affine connection Γ^i_{kl} in a pure Riemannian space by the relation (ref. 16, 5.9)

$$\Delta^i{}_{kl} = \Gamma^i_{kl} + \gamma^i{}_{kl}. \tag{2.25}$$

The covariant derivatives of the second kind of a vector field with components A^i and A_k are

$$\left.\begin{aligned} A^i{}_{|l} &= A^i{}_{,l} + \Delta^i{}_{kl}A^k = A^i{}_{;l} + \gamma^i{}_{kl}A^k \\ A_{k|l} &= A_{k,l} - \Delta^i{}_{kl}A_i = A_{k;l} - \gamma^i{}_{kl}A_i \end{aligned}\right\} \tag{2.26}$$

with obvious generalizations for tensors of higher rank.

When (2.11) is used in the usual expression for the curvature tensor, R^i_{klm} appears as a function of the tensor γ_{ikl} and its first order covariant derivatives. In (ref. 16, D. 6) it is given in terms of derivatives of the second kind. In terms of the usual derivatives we have

$$R^i{}_{klm} = \gamma^i{}_{km;l} - \gamma^i{}_{kl;m} + \gamma^i{}_{rl}\gamma^r{}_{km} - \gamma^i{}_{rm}\gamma^r{}_{kl}. \tag{2.27}$$

Further, if Φ_k is the vector obtained by contraction of $\gamma^i{}_{kl}$

$$\Phi_k = \gamma^i{}_{ki} = -\gamma_k{}^i{}_i = -\overset{a}{h}_k \underset{a}{h}{}^i{}_{;i}, \tag{2.28}$$

the curvature scalar R can be written in the form

$$R = -\frac{2}{\sqrt{-g}}(\sqrt{-g}\,\Phi^r)_{,r} + \gamma_{rst}\gamma^{tsr} - \Phi_r\Phi^r. \tag{2.29}$$

Here we have used (ref. 16, A. 5–A. 7), (2.28) and

$$\gamma_{rst}\gamma^{tsr} = \overset{a}{h}{}^r{}_{;s}\underset{a}{h}{}^s{}_{;r} \tag{2.30}$$

following from (2.18) and (2.17).

For a given tetrad field $\underset{a}{}{}^i$ the metric field is uniquely given by (2.11), (2.16). However a given metric g^{ik} does not determine the tetrad field completely; for any Lorentz rotation of the tetrads leads to a new set of tetrads $\underset{a}{\lambda}{}^i$ which also satisfy all the relations (2.2–16). Arbitrary point dependent Lorentz rotations of the tetrads are given by

$$\underset{a}{\lambda}{}^i = \underset{a}{\overset{b}{\Omega}}(x)\underset{b}{h}{}^i, \quad \overset{a}{\lambda}{}^i = {}^a\underset{b}{\Omega}(x)\overset{b}{h}{}^i \tag{2.31}$$

where the rotation coefficients $\underset{a}{\Omega}^{b}(x)$ and the functions

$$\overset{a}{\underset{b}{\Omega}}(x) = \varepsilon_{a)}\,\varepsilon_{b)}\,\underset{a}{\Omega}^{b}(x) \tag{2.32}$$

are scalars satisfying the Lorentz conditions

$$\underset{a}{\Omega}^{c}\,\overset{b}{\underset{c}{\Omega}} = \overset{c}{\underset{a}{\Omega}}\,\underset{c}{\Omega}^{b} = \delta^{a}_{b}, \tag{2.33}$$

Hence

$$\underset{a}{\lambda}^{i}\overset{a}{\lambda}^{k} = \underset{a}{\Omega}^{c}\,\overset{a}{\underset{b}{\Omega}}\,\underset{c}{h}^{i}\overset{b}{h}^{k} = \delta^{c}_{b}\underset{c}{h}^{i}\overset{b}{h}^{k} = g^{ik} \tag{2.34}$$

for arbitrary functions $\underset{a}{\Omega}^{b}(x)$ satisfying (2.33).

Since the Lorentz group is a 6-parametric group, the general solution $\underset{a}{h}^{i}(x)$ of (2.16) for a given metric contains six arbitrary functions. Therefore, besides ten equations determining the metric as in EGRG, the field equations in the present theory must contain six further equations. It should be noticed, however, that a Lorentz rotation (2.31) with constant $\underset{a}{\Omega}^{b}$ does not change neither g_{ik} nor γ_{ikl}. In this case $\underset{a}{\lambda}^{i}$ and $\underset{a}{h}^{i}$ define a space-time with identical curvature and torsion, i.e. the two tetrad lattices describe the same physical situation. However, apart from a constant Lorentz rotation the tetrad field must be completely determined by the field equations.

The situation in special relativity is characterized by a vanishing torsion, i.e.

$$\gamma_{ikl} = 0 \tag{2.35}$$

which by (2.27) entails a vanishing curvature:

$$R_{iklm} = 0 \tag{2.36}$$

This equation allows the introduction of a pseudo-Cartesian system of coordinates with

$$g_{ik} = \eta_{ik} = g^{ik}. \tag{2.37}$$

Then the equation (2.35) gives

$$\underset{a}{h}^{i}{}_{;k} = \underset{a}{h}^{i}{}_{,k} = 0 \tag{2.38}$$

i.e. the $\underset{a}{h}^{i}$ are constant in this system of coordinates and by a suitable constant Lorentz rotation we can make

$$\underset{a}{h}^{i} = \delta^{i}_{a}. \tag{2.39}$$

For an insular matter system, (2.37) and (2.39) can be chosen as the limiting values of g_{ik} and $\underset{a}{h}^{i}$ for spatial distances $r \to \infty$.

3. The General Form of the Field Equations

In accordance with **D** we assume that the field equations are derivable from a Lagrangean principle. The gravitational part $\mathfrak{L} = \sqrt{-g}\, L$ of the Lagrangean density must be a scalar density under coordinate transformations, i.e. L is a scalar constructed from the gravitational potentials $\underset{a}{h}{}^i$ and their derivatives of the first order

$$L = L(\underset{a}{h}{}^i, \underset{a}{h}{}^i{}_{,k}) \tag{3.1}$$

(higher order derivatives in (3.1) would violate condition **E**). Since a constant rotation of the tetrads shall have no physical effect we have to require that L is invariant also under the group of constant Lorentz rotations. According to (2.21) the tensor γ_{ikl} is a linear homogeneous function of the first order derivatives of the tetrads and it is invariant under constant Lorentz rotations. Furthermore it is essentially the only tensor with these properties. Therefore L must be a scalar constructed from the γ_{ikl} and the metric tensor g_{ik}.

The variation of the Lagrangean integral under arbitrary variations $\delta \underset{a}{h}{}^i$ that vanish at the boundary of the region of integration is

$$\left.\begin{aligned} \delta\int \mathfrak{L} dx &= \delta \int L\sqrt{-g}\, dx \\ &= \int \frac{\delta \mathfrak{L}}{\delta \underset{a}{h}{}^i} \delta \underset{a}{h}{}^i dx, \end{aligned}\right\} \tag{3.2}$$

where

$$\frac{\delta \mathfrak{L}}{\delta \underset{a}{h}{}^i} = \frac{\partial \mathfrak{L}}{\partial \underset{a}{h}{}^i} - \left(\frac{\partial \mathfrak{L}}{\partial \underset{a}{h}{}^i{}_{,l}}\right)_{,l} \tag{3.3}$$

is the variational derivative of $\mathfrak{L}$ with respect to $\underset{a}{h}{}^i$. (3.2) may also be written

$$\delta\int \mathfrak{L}\, dx = \int V_{ik} \overset{a}{h}{}^k \delta \underset{a}{h}{}^i \sqrt{-g}\, dx, \tag{3.4}$$

where V_{ik} is the tensor

$$V_{ik} = \frac{1}{\sqrt{-g}} \frac{\delta \mathfrak{L}}{\delta \underset{a}{h}{}^i} \underset{a}{h}{}_k. \tag{3.5}$$

From (2.16) we get for the variation of g^{ik} corresponding to the variation $\delta \underset{a}{h}{}^i$

$$\left.\begin{aligned} \delta g^{ik} &= \underset{a}{h^i}\delta \overset{a}{h^k} + \overset{a}{h^k}\delta \underset{a}{h^i} \\ &= \overset{a}{h^i}\delta \underset{a}{h^k} + \overset{a}{h^k}\delta \underset{a}{h^i} = \delta g^{ki}. \end{aligned}\right\} \quad (3.6)$$

If we define a quantity δf^{ik} by

$$\delta f^{ik} = \overset{a}{h^i}\delta \underset{a}{h^k} - \overset{a}{h^k}\delta \underset{a}{h^i} = -\delta f^{ki} \quad (3.7)$$

we have

$$\overset{a}{h^k}\delta \underset{a}{h^i} = \tfrac{1}{2}(\delta g^{ik} - \delta f^{ik}). \quad (3.8)$$

Thus (3.4) may be written

$$\delta \int \mathfrak{L} dx = \int (S_{ik}\delta g^{ik} + F_{ik}\delta f^{ik})\sqrt{-g}\, dx \quad (3.9)$$

with

$$\left.\begin{aligned} S_{ik} &= \tfrac{1}{2} V_{(ik)} = S_{ki} \\ F_{ik} &= -\tfrac{1}{2} V_{[ik]} = -F_{ki}. \end{aligned}\right\} \quad (3.10)$$

As usual $V_{(ik)}$ and $V_{[ik]}$ denote the symmetrical and antisymmetrical combinations, respectively, i.e.

$$\left.\begin{aligned} V_{(ik)} &= \tfrac{1}{2}(V_{ik} + V_{ki}) \\ V_{[ik]} &= \tfrac{1}{2}(V_{ik} - V_{ki}). \end{aligned}\right\} \quad (3.11)$$

By well-known methods we can derive an identity involving S_{ik} and F_{ik} from the invariance of the Lagrangean integral $\int \mathfrak{L} dx$ under arbitrary infinitesimal coordinate transformations

$$\bar{x}^i = x^i + \xi^i(x). \quad (3.12)$$

The corresponding "local" variations of g^{ik} and $\underset{a}{h^i}$ are

$$\left.\begin{aligned} \delta g^{ik} &= g^{il}\xi^k{}_{,l} + g^{lk}\xi^i{}_{,l} - g^{ik}{}_{,l}\xi^l \\ \delta \underset{a}{h^i} &= \underset{a}{h^l}\xi^i{}_{,l} - \underset{a}{h^i}{}_{,l}\xi^l \end{aligned}\right\} \quad (3.13)$$

and, by (3.7) and (2.16),

$$\left.\begin{aligned} \delta f^{ik} &= g^{il}\xi^k{}_{,l} - g^{kl}\xi^i{}_{,l} \\ &\quad + (\overset{a}{h^k}\underset{a}{h^i}{}_{,l} - \overset{a}{h^i}\underset{a}{h^k}{}_{,l})\,\xi^l. \end{aligned}\right\} \quad (3.14)$$

Introduction of (3.13) and (3.14) into (3.9) gives after partial integrations

$$\delta \int \mathfrak{L} dx = 2 \int \{ - S_i{}^k{}_{;k} + F_i{}^k{}_{;k} - F^{kl} \gamma_{kli} \} \xi^i \sqrt{-g}\, dx = 0. \qquad (3.15)$$

for arbitrary $\xi^i(x)$ vanishing at the boundary. Hence the identity

$$S_i{}^k{}_{;k} \equiv F_i{}^k{}_{;k} - F^{kl} \gamma_{kli}. \qquad (3.16)$$

Let $\mathfrak{L}_m$ denote the usual Lagrangean density of a macroscopic body, which in addition to the matter variables depends on the metric tensor only. Then the variation of the gravitational variables gives

$$\delta \int \mathfrak{L}_m dx = \int T_{ik} \delta g^{ik} \sqrt{-g}\, dx, \qquad (3.17)$$

where T_{ik} is the energy-momentum tensor of the matter. By means of (3.9) and (3.17) the Lagrangean principle for the gravitational field in the presence of matter is

$$\left.\begin{aligned} \delta \int (\mathfrak{L} + \mathfrak{L}_m)\, dx &= \delta \int (L + L_m) \sqrt{-g}\, dx \\ &= \int \{ S_{ik} + T_{ik}) \delta g^{ik} + F_{ik} \delta f^{ik} \} \sqrt{-g}\, dx = 0 \end{aligned}\right\} \qquad (3.18)$$

for arbitrary variations $\delta \underset{a}{h^i}$ of the 16 functions $\underset{a}{h^i}$. These variations may be written

$$\delta \underset{a}{h^i} = \underset{ab}{\varepsilon} \overset{b}{h^i} \qquad (3.19)$$

where the

$$\underset{ab}{\varepsilon}(x) = \delta \underset{a}{h^i} \cdot \underset{b}{h_i}(x) \qquad (3.20)$$

are 16 independent infinitesimal functions. Writing $\underset{ab}{\varepsilon}$ as a sum of a symmetrical and antisymmetrical part

$$\underset{ab}{\varepsilon} = \underset{ab}{\sigma} + \underset{ab}{\omega}, \qquad \underset{ab}{\sigma} = \underset{(ab)}{\varepsilon} = \underset{ba}{\sigma}, \qquad \underset{ab}{\omega} = \underset{[ab]}{\varepsilon} = -\underset{ba}{\omega}, \qquad (3.21)$$

we get

$$\delta \underset{a}{h^i} = \delta_{(d)} \underset{a}{h^i} + \delta_{(r)} \underset{a}{h^i} \qquad (3.22)$$

with

$$\left.\begin{aligned} \delta_{(d)} \underset{a}{h^i} &= \underset{ab}{\sigma} \overset{b}{h^i} \\ \delta_{(r)} \underset{a}{h^i} &= \underset{ab}{\omega} \overset{b}{h^i}. \end{aligned}\right\} \qquad (3.23)$$

The latter variation is obviously an infinitesimal Lorentz rotation of the type (2.31), (2.33) with

$$\underset{a}{\Omega}^{b} = \delta_a^b + \underset{a}{\omega}^{b} = \delta_a^b + \varepsilon_{b)} \underset{ab}{\omega}, \tag{3.24}$$

which leaves g^{ik} unchanged. In fact we get from (3.6), (3.7) and (3.23)

$$\left.\begin{aligned} \delta_{(r)} g^{ik} &= (\underset{ab}{\omega} + \underset{ba}{\omega}) \overset{a}{h^i} \overset{b}{h^k} = 0 \\ \delta_{(r)} f^{ik} &= 2 \underset{ab}{\omega} \overset{a}{h^i} \overset{b}{h^k} \end{aligned}\right\} \tag{3.25}$$

and

$$\left.\begin{aligned} \delta_{(d)} g^{ik} &= 2 \underset{ab}{\sigma} \overset{a}{h^i} \overset{b}{h^k} \\ \delta_{(d)} f^{ik} &= 0\,. \end{aligned}\right\} \tag{3.26}$$

According to (3.22), (3.25), (3.26) a general variation $\delta \underset{a}{h^i}$ is composed of 10 independent "dilations" $\delta_{(d)} \underset{a}{h^i}$ for which $\delta f^{ik} = 0$ and 6 independent "rotations" $\delta_{(r)} \underset{a}{h^i}$ for which $\delta g^{ik} = 0$. Therefore the variational principle (3.18) leads to the field equations

$$S_{ik} + T_{ik} = 0\,, \tag{3.27}$$

$$F_{ik} = 0. \tag{3.28}$$

The 10+6 field equations (3.27), (3.28) determine the 16 tetrad functions apart from arbitrary constant Lorentz rotations. From (3.27) and the identity (3.16) we get

$$T_i{}^k{}_{;k} = -S_i{}^k{}_{;k} = -F_i{}^k{}_{;k} + F^{kl}\gamma_{kli} = 0$$

on account of (3.28), i.e. the usual conservation law (1.2) is a consequence of the field equations as in Einstein's theory.

With an arbitrary L constructed from the γ_{ikl} and g_{ik} we have thus a formalism in which all the essential properties **A**–**E** are valid. In particular the equivalence principle is valid exactly and the world line of a freely falling particle is a geodesic in the space with the metric (2.11), but the metric determined by (3.27), (3.28) will of course in general be different from the metric following from Einstein's field equations. Moreover a theory of this type will give a more satisfactory expression for the energy-momentum complex, since the necessary conditions formulated in ref. 18 are satisfied in the present formalism.

4. The Choice of Lagrangean

The arbitraryness in the choice of Lagrangean is decisively limited by the essential requirement that the theory must give the same results as EGRG for the gravitational phenomena inside the solar system. Since L is an invariant constructed from the γ_{ikl} and g_{ik} the simplest possible independent expressions are

$$\left.\begin{aligned} L^{(1)} &= \Phi_r \Phi^r, \quad L^{(2)} = \gamma_{rst}\gamma^{rst}, \\ L^{(3)} &= \gamma_{rst}\gamma^{tsr} \end{aligned}\right\} \tag{4.1}$$

where Φ_k is the vector (2.28)

$$\Phi_k = \gamma^i{}_{ki}. \tag{4.2}$$

On account of (2.21) the expressions $L^{(\nu)}$ in (4.1) are homogeneous functions of the first order derivatives $\underset{a}{h}{}^r{}_{,t}$ of degree 2. The next simplest algebraic expressions are obviously of degree 4 and there are not less than twelve different independent expressions of this type.

In the simplest case L is a linear combination of the quantities (4.1)

$$\mathfrak{L} = \sum_{\nu=1}^{3} \alpha_\nu \mathfrak{L}^{(\nu)}, \quad \mathfrak{L}^{(\nu)} = \sqrt{-g}\, L^{(\nu)}. \tag{4.3}$$

For each ν we have an equation of the form (3.9)

$$\delta \int \mathfrak{L}^{(\nu)} dx = \int (S^{(\nu)}_{ik}\, \delta g^{ik} + F^{(\nu)}_{ik}\, \delta f^{ik}) \sqrt{-g}\, dx, \tag{4.4}$$

and with (4.3) we obtain

$$\left.\begin{aligned} S_{ik} &= \sum_{\nu=1}^{3} \alpha_\nu S^{(\nu)}_{ik} \\ F_{ik} &= \sum_{\nu=1}^{3} \alpha_\nu F^{(\nu)}_{ik}. \end{aligned}\right\} \tag{4.5}$$

A lengthy but elementary calculation gives the following explicit expressions for $S^{(\nu)}_{ik}$ and $F^{(\nu)}_{ik}$:

$$\left.\begin{aligned} S^{(1)}_{ik} &= \tfrac{1}{2}(\Phi_{i;k} + \Phi_{k;\ }) - \tfrac{1}{2}\Phi_l(\gamma^l{}_{ik} + \gamma^l{}_{ki}) - g_{ik}(\Phi^l{}_{;l} + \tfrac{1}{2}\Phi_l\Phi^l), \\ S^{(2)}_{ik} &= \gamma^l{}_{ik;l} + \gamma^l{}_{ki;l} + \gamma_{rsi}\gamma^{rs}{}_k - \tfrac{1}{2} g_{ik}\gamma_{rst}\gamma^{rst}, \\ S^{(3)}_{ik} &= \tfrac{1}{2}[\gamma^l{}_{ik;l} + \gamma^l{}_{ki;l}] - \tfrac{1}{2}[\gamma_{rsi}\gamma_k{}^{rs} + \gamma_{rsk}\gamma_i{}^{rs}] - \tfrac{1}{2} g_{ik}\gamma_{rst}\gamma^{tsr} \end{aligned}\right\} \tag{4.6}$$

$$\left.\begin{aligned} F^{(1)}_{ik} &= F^{(3)}_{ik} = \tfrac{1}{2}\left[\Phi_{i,k} = \Phi_{k,i} - \Phi_l(\gamma^l{}_{ik} - \gamma^l{}_{ki})\right] \\ F^{(2)}_{ik} &= -\gamma_{ik}{}^l{}_{;l}. \end{aligned}\right\} \tag{4.7}$$

The Langrangean density

$$\mathfrak{L}^{(0)} = \mathfrak{L}^{(3)} - \mathfrak{L}^{(1)} = \sqrt{-g}\,(\gamma_{rst}\gamma^{tsr} - \Phi_r\Phi^r) \tag{4.8}$$

has the remarkable property that $\int \mathfrak{L}^{(0)} dx$ is invariant under arbitrary infinitesimal Lorentz rotations of the tetrads; for we have, since $F^{(1)}_{ik} = F^{(3)}_{ik}$

$$\delta \int \mathfrak{L}^{(0)} dx = \int (S^{(3)}_{ik} - S^{(1)}_{ik})\,\delta g^{ik}\sqrt{-g}\,dx. \tag{4.9}$$

This is in accordance with the fact shown in (ref. 16, Appendix A), that $\mathfrak{L}^{(0)}$ is equal to the Lagrangean density $\sqrt{-g}\,R$ in Einstein's theory, apart from a usual divergence which can be disregarded in the variations considered. Thus

$$\delta \int \mathfrak{L}^{(0)}\, dx = \delta \int R\sqrt{-g}\, dx = \int G_{ik}\,\delta g^{ik}\sqrt{-g}\, dx \tag{4.10}$$

where G_{ik} is the Einstein tensor in (1.3). A comparison of (4.9) and (4.10) gives

$$G_{ik} = S^{(3)}_{ik} - S^{(1)}_{ik} \tag{4.11}$$

in accordance with (ref. 16, D. 7, D. 8).

We shall now choose the constants α_ν such that our theory gives the same results as EGRG in the linear approximation of weak fields. In a suitable system of coordinates we have in this case

$$g_{ik} = \eta_{ik} + y_{ik}, \tag{4.12}$$

where the small quantities y_{ik} satisfy the de Donder relations

$$\varepsilon_k y_{ik,k} = \tfrac{1}{2} y_{,i}, \quad y = \varepsilon_k y_{kk}. \tag{4.13}$$

Then, neglecting terms of the second order in y_{ik}, Einstein's equations (1.3) reduce to

$$\tfrac{1}{2}(\Box y_{ik} - \tfrac{1}{2}\eta_{ik}\Box y) = -\varkappa T_{ik}, \tag{4.14}$$

where

$$\Box = \varepsilon_k \frac{\partial^2}{\partial x^{k2}} \tag{4.15}$$

is the usual d'Alembertian.

In the same approximation the tetrads

$$\underset{a}{h}_i = \eta_{ai} + \tfrac{1}{2} y_{ai} \tag{4.16}$$

obviously satisfy (2.11) with g_{ik} given by (4.12), and from (4.16) we get using (4.13)

$$\left.\begin{aligned} \gamma_{ikl} &= \tfrac{1}{2}(y_{kl,i} - y_{il,k}) \\ \Phi_k &= -\tfrac{1}{4} y_{,k}. \end{aligned}\right\} \quad (4.17)$$

These quantities are small of 1. order. Therefore, neglecting terms of 2. order and using (4.13), the equations (4.6) and (4.7) give

$$\left.\begin{aligned} S_{ik}^{(1)} &= \tfrac{1}{2}(\Phi_{i,k} + \Phi_{k,i}) - \eta_{ik}\varepsilon_l \Phi_{l,l} \\ &= \tfrac{1}{4}(\eta_{ik}\Box y - y_{,i,k}), \\ S_{ik}^{(2)} &= 2 S_{ik}^{(3)} = \varepsilon_l \gamma_{lik,l} + \varepsilon_l \gamma_{lki,l} \\ &= \Box y_{ik} - \tfrac{1}{2} y_{,i,k}, \end{aligned}\right\} \quad (4.18)$$

$$\left.\begin{aligned} F_{ik}^{(1)} &= F_{ik}^{(3)} = \tfrac{1}{2}(\Phi_{i,k} - \Phi_{k,i}) = 0 \\ F_{ik}^{(2)} &= -\varepsilon_l \gamma_{ikl,l} = -\tfrac{1}{2}(\varepsilon_l y_{kl,i,l} - \varepsilon_l y_{il,k,l}) \\ &= -\tfrac{1}{4}(y_{,k,i} - y_{,i,k}) = 0. \end{aligned}\right\} \quad (4.19)$$

From the latter equations and (4.5) we see that the expressions (4.16) satisfy the field equations (3.28):

$$F_{ik} = 0, \quad (4.20)$$

and it can be shown (ref. 16, § 4) that (4.16) are the only expressions satisfying (4.20), apart of course from physically unimportant constant Lorentz rotations.

With (4.18) we get for S_{ik} in (4.5)

$$S_{ik} = (2\alpha_2 + \alpha_3)\tfrac{1}{2}\Box y_{ik} + \frac{\alpha_1}{4}\eta_{ik}\Box y - \tfrac{1}{4}(\alpha_1 + 2\alpha_2 + \alpha_3) y_{,i,k}. \quad (4.21)$$

When (4.21) is introduced into the field equations (3.27), it is seen that the latter equations be identical with the linear Einstein equations (4.14), if we choose

$$\alpha_1 = -\frac{1}{\varkappa}, \quad \alpha_2 = \frac{\lambda}{\varkappa}, \quad \alpha_3 = \frac{1}{\varkappa}(1 - 2\lambda) \quad (4.22)$$

with λ equal to an arbitrary dimensionless constant. With these values for the α_ν we get from (4.5), (4.6), (4.7) and (4.11)

$$\left.\begin{aligned} \varkappa S_{ik} &= -S_{ik}^{(1)} + \lambda S_{ik}^{(2)} + S_{ik}^{(3)} - 2\lambda S_{ik}^{(3)} \\ &= G_{ik} + \lambda (S_{ik}^{(2)} - 2S_{ik}^{(3)}), \\ F_{ik} &= -\frac{\lambda}{\varkappa}(2F_{ik}^{(3)} - F_{ik}^{(2)}). \end{aligned}\right\} \quad (4.23)$$

For $\lambda = 0$ the present theory is identical with Einstein's theory, but for $\lambda \neq 0$ the field equations (3.27), (3.28) take the form

$$G_{ik} + H_{ik} = -\varkappa T_{ik}, \quad (4.24)$$

$$2F_{ik}^{(3)} - F_{ik}^{(2)} = \Phi_{i,k} - \Phi_{k,i} - \Phi_l(\gamma^l{}_{ik} - \gamma^l{}_{ki}) + \gamma_{ik}{}^l{}_{;l} = 0, \quad (4.25)$$

$$H_{ik} = \lambda[\gamma_{rsi}\gamma^{rs}{}_k + \gamma_{rsi}\gamma_k{}^{rs} + \gamma_{rsk}\gamma_i{}^{rs} + g_{ik}(\gamma_{rst}\gamma^{tsr} - \tfrac{1}{2}\gamma_{rst}\gamma^{rst})]. \quad (4.26)$$

The equations (4.25) are independent of the choice of λ. On the other hand the term H_{ik}, by which (4.24) deviates from Einstein's field equations (1.3) increases with λ, which can be taken of order 1 without destroying the first order agreement with Einstein's theory in the weak field case. One might hope, therefore, that the metric obtained as solution of (4.24), (4.25) would be quite different from the solution of (1.3) in the case of strong fields and that it be free of singularities. In the next section we shall investigate this point by considering the case of a spherically symmetric system.

5. The Spherically Symmetric Case

In the case of a *static* spherically symmetric system the equations (4.24), (4.25) are most easily solved if we use a system of isotropic coordinates $x^i = \{x^\iota, ct\}$. Here the metric is of the form

$$\left.\begin{aligned} g_{ik} &= g_{ii)}\delta_{ik}, \quad g^{ik} = \frac{1}{g_{ii)}}\delta_{ik}, \\ g_{ii} &= \{a, a, a, -b\}, \end{aligned}\right\} \quad (5.1)$$

where a and b are functions of $r = x^\iota x^\iota$ only. A possible set of tetrads in accordance with (2.11) and (5.1) is

$$\left.\begin{aligned} \underset{a}{h^i} &= \frac{1}{\sqrt{|g_{aa)}|}}\delta_a{}^i \\ \underset{a}{h_i} &= g_{ii)}\underset{a}{h^i} = \varepsilon_{a)}\sqrt{|g_{aa}|}\,\delta_{ai} \end{aligned}\right\} \quad (5.2)$$

from which we get the following expression for the tensor (2.18) and the vector (4.2) (see ref. 16, B. 4, B. 8)

$$\left.\begin{aligned} \gamma_{ikl} &= \frac{g'_{ll)}(r)}{2}\left(n_i\delta_{kl} - n_k\delta_{il}\right) \\ n_i &= \frac{\partial r}{\partial x^i} = \left\{\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}, 0\right\} \end{aligned}\right\} \qquad (5.3)$$

and

$$\Phi_k = -(ln\, a\sqrt{b})_{,k} = -(ln\, a\sqrt{b})'\, n_k. \qquad (5.4)$$

By calculating the functions (4.7) with (5.3), (5.4) and (5.1) one finds (see the corresponding calculations in ref. 16, Appendix B)

$$F_{ik}^{(1)} = F_{ik}^{(2)} = F_{ik}^{(3)} = 0, \quad F_{ik} = 0 \qquad (5.5)$$

Thus the tetrads (5.2) satisfy the field equations (4.25), and it can be shown that (5.2) are the only tetrads satisfying these equations, again apart from constant rotations of the tetrads.

Using (5.3) and (5.1) we get for the different terms in (4.26)

$$\left.\begin{aligned} \gamma_{rsi}\gamma^{rs}{}_k &= -2\gamma_{rsi}\gamma_k{}^{rs} = -2\gamma_{rsk}\gamma_i{}^{rs} \\ &= \frac{(g_{ii)}')^2}{2\,a\,g_{ii}}\,\delta_{ik} - \frac{a'^2}{2a^2}\,n_i n_k \\ \gamma_{rst}\gamma^{rst} &= 2\gamma_{rst}\gamma^{tsr} = \frac{a'^2}{a^3} + \frac{b'^2}{2\,ab^2} \end{aligned}\right\} \qquad (5.6)$$

i.e.

$$H_{ik} = 0. \qquad (5.7)$$

Thus, in the static spherically symmetric case the equations (4.24) have the same solutions as Einstein's equations (1.3). In the empty space outside the matter they lead to the following equations for $a(r)$ and $b(r)$:

$$\left.\begin{aligned} a'' + \frac{2a'}{r} - \frac{3}{4}\frac{a'^2}{a} &= 0 \\ \left(\frac{a'}{r} + \frac{2}{r}\right)\frac{b'}{b} + \frac{2a'}{ra} + \frac{a'^2}{2\,a} &= 0 \end{aligned}\right\} \qquad (5.8)$$

with the well-known solutions

$$a = (1+\alpha/4r)^4, \;\; b = \frac{(1-\alpha/4r)^2}{(1+\alpha/4r)^2}. \tag{5.9}$$

The functions $a(r)$ and $b(r)$ in (5.9) are everywhere positive except at the Schwarzschild distance $r = \alpha/4$ where $b(r)$ has the minimum value zero. It would seem that only a small change of the equations (5.8) is necessary to make the minimum value of $b(r)$ positive and thus remove the singularity.

So far we have only considered the static case. As an important example of a time-dependent spherical system we shall now consider the case of the non-static homogeneous isotropic universe. In suitable coordinates the metric has the form given by Robertson and Walker, i.e.

$$\left.\begin{aligned} g_{ik} &= g_{ii)}\,\delta_{ik}, \;\; g_{ii} = \{a, a, a, -1\} \\ a &= \frac{R(t)^2}{\psi(r)^2}, \;\; \psi(r) = 1+\zeta r^2/4 \\ \zeta &= \begin{cases} 1 \\ 0 \\ -1 \end{cases}. \end{aligned}\right\} \tag{5.10}$$

With tetrads of the form (5.2) we get in this case

$$\gamma_{ikl} = \begin{cases} 0 \quad \text{for} \quad l = 4 \\ \frac{1}{2}[a,_i \delta_{k\lambda} - a,_k \delta_{i\lambda}] \;\; \text{for} \;\; l = \lambda \end{cases} \tag{5.11}$$

and

$$\Phi_k = \gamma^i{}_{ki} = \left(ln \frac{\Psi^2}{R^3}\right)_{,k}. \tag{5.12}$$

Calculating the tensors (4.7) with (5.11) and (5.12) we obtain

$$F^{(1)}_{ik} = F^{(3)}_{ik} = \tfrac{1}{2} F^{(2)}_{ik} = (ln R),_i (ln \Psi),_k - (ln R),_k (ln \Psi),_i, \tag{5.13}$$

which shows that the field equations (4.25) are satisfied with $\underset{a}{h^i}$ given (5.2). Further we get for the different terms in (4.26)

$$\left.\begin{aligned} \gamma_{rsi}\gamma^{rs}{}_k &= -2\gamma_{rsi}\gamma_k{}^{rs} = -2\gamma_{rsk}\gamma_i{}^{rs} \\ &= \begin{cases} 0 \;\; \text{for} \;\; i = 4 \;\; \text{or} \;\; k = 4 \\ \frac{2}{\Psi^2}[(\Psi'^2 - \dot{R}^2)\delta_{\iota\varkappa} - \Psi'^2 n_\iota n_\varkappa] \;\; \text{for} \;\; i = \iota, k = \varkappa \end{cases} \end{aligned}\right\} \tag{5.14}$$

Thus, H_{ik} vanishes also in this case, and the metric following from the present formalism is again given by the Friedman solution which has a singularity in the far past and for $\zeta = 1$ also in the far future.

As we have seen the simple Lagrangean density

$$\mathfrak{L} = \frac{1}{\varkappa}[\mathfrak{L}^{(0)} + \lambda(\mathfrak{L}^{(2)} - 2\mathfrak{L}^{(3)})], \tag{5.15}$$

which leads to the field equations (4.24)–(4.26), does not solve our problem. However, as mentioned before there is a large variety of possible expressions $\mathfrak{L}^{(4)}$ of degree 4, and with

$$\mathfrak{L} = \frac{1}{\varkappa}\mathfrak{L}^{(0)} + \mathfrak{L}^{(4)} \tag{5.16}$$

the variational principle leads to equations of the form (4.24) with a non-vanishing H_{ik} in the static spherically symmetric case. Instead of (5.8) we get then

$$\left.\begin{aligned} a'' + \frac{2a'}{r} - \frac{3}{4}\frac{a'^2}{a} &= f \\ \left(\frac{a'}{a} + \frac{2}{r}\right)\frac{b'}{b} + \frac{2a'}{ra} + \frac{a'^2}{2a} &= g \end{aligned}\right\} \tag{5.17}$$

where f and g in general are algebraic functions of a, b, a', b', a'' and b'' depending on the choice of $\mathfrak{L}^{(4)}$. Besides terms of degree 4, which in the case of weak fields give contributions to H_{ik} that are small of the third order, we may in $\mathfrak{L}^{(4)}$ also include terms of the type $\frac{1}{\varkappa}\sum_{\nu=1}^{3}\lambda_\nu \mathfrak{L}^{(\nu)}$ with sufficiently small dimensionless constants λ_ν. It would be surprising if not one of the many possible Lagrangeans would lead to equations (5.17) with everywhere positive solutions $a(r)$, $b(r)$. On the contrary one could rather fear that there are too many Lagrangeans that have singularity free solutions, in which case it would be difficult to obtain a uniquely determined theory without a new guiding principle.

Conclusion

In the present paper we have not arrived at a definite formalism which can replace Einstein's precise equations. We have shown only that the breakdown of physics predicted by Hawking on the basis of Einstein's theory does not seem to be inevitable. If we admit that the fundamental gravitational field variables are tetrad fields, the way is open for generalizations of Einstein's theory which retain all the satisfactory features **A**–**E** as well as the experimentally and observationally verified results of EGRG. At the same time such a formalism allows a more satisfactory treatment of the energy-momentum complex, in particular as regards the question of the localizability of the energy. It still remains to be seen if the Lagrangean can be chosen in such a way that the field equations in all cases have non-singular solutions.

References

1) R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).
S. W. Hawking, Proc. Roy. Soc. A **294**, 511 (1966); **295**, 490 (1966); **300**, 187 (1967).
S. W. Hawking and R. Penrose, Proc. Roy. Soc. A **314**, 529 (1970).
2) See f. inst. E. M. Lifshitz and I. M. Khalatnikov, Adv. in Phys. **12**, 185 (1963).
3) S. W. Hawking, Nature **248**, 30 (1974); Commun. Math. Phys. **17** (2), 174.
4) R. M. Wald, On Particle Creation by Black Holes (Preprint, University of Chicago, 1975).
5) L. Parker, Probability Distribution of Particles Created by a Black Hole (Preprint, University of Wisconsin Milwaukee, 1975).
6) S. W. Hawking, Fundamental Breakdown of Physics in Gravitational Collapse. (Orange Aid Preprint, OAP-420, California Institute of Technology, 1975).
7) A. Einstein, The Meaning of Relativity, Princeton University Press 1953, p. 129.
8) See for instance C. Møller, Measurements in General Relativity and the Principle of Relativity in Problems of Theoretical Physics, Ivanenko Festschrift, Moscow 1976.
9) C. Møller, On the Behaviour of Physical Clocks in the Vicinity of Singularities of a Gravitational Field. Report Ettore Majorana School of Physics, Erice, March 1975.
10) P. J. Role, R. Krotkov and R. H. Dicke, Ann. Phys. **26**, 442 (1964).
11) V. B. Braginsky and V. I. Panov, Zh. Eksp. Theor. Fiz. **61**, 873 (1971).
12) H. Reissner, Annln. Phys. **50**, 106 (1916).
13) H. Weyl, Annln. Phys. **54**, 117 (1917).
14) See f. inst. C. Møller, The Theory of Relativity, 2. Edition, Oxford 1972, § 4, 6, p. 104.
15) R. V. Pound and G. A. Rebka, Phys. Rev. Lett. **4**, 337 (1960).
R. V. Pound and J. L. Snider, Phys. Rev. Lett. **13**, 539 (1964).
16) C. Møller, Mat. Fys. Skr. Dan. Vid. Selsk. **1**, no 10 (1961).
17) C. Pellegrini and J. Plebanski, Mat. Fys. Skr. Dan. Vid. Selsk. **2**, no 4 (1963).
18) C. Møller, Mat. Fys. Medd. Dan. Vid. Selsk. **35**, no. 3 (1966).
19) R. Weitzenböck, Invariantentheorie p. 317ff., Groningen 1923; Sber. preuss. Akad. Wiss. 466 (1928).

Indleveret til Selskabet September 1977.
Færdig fra trykkeriet Januar 1978.